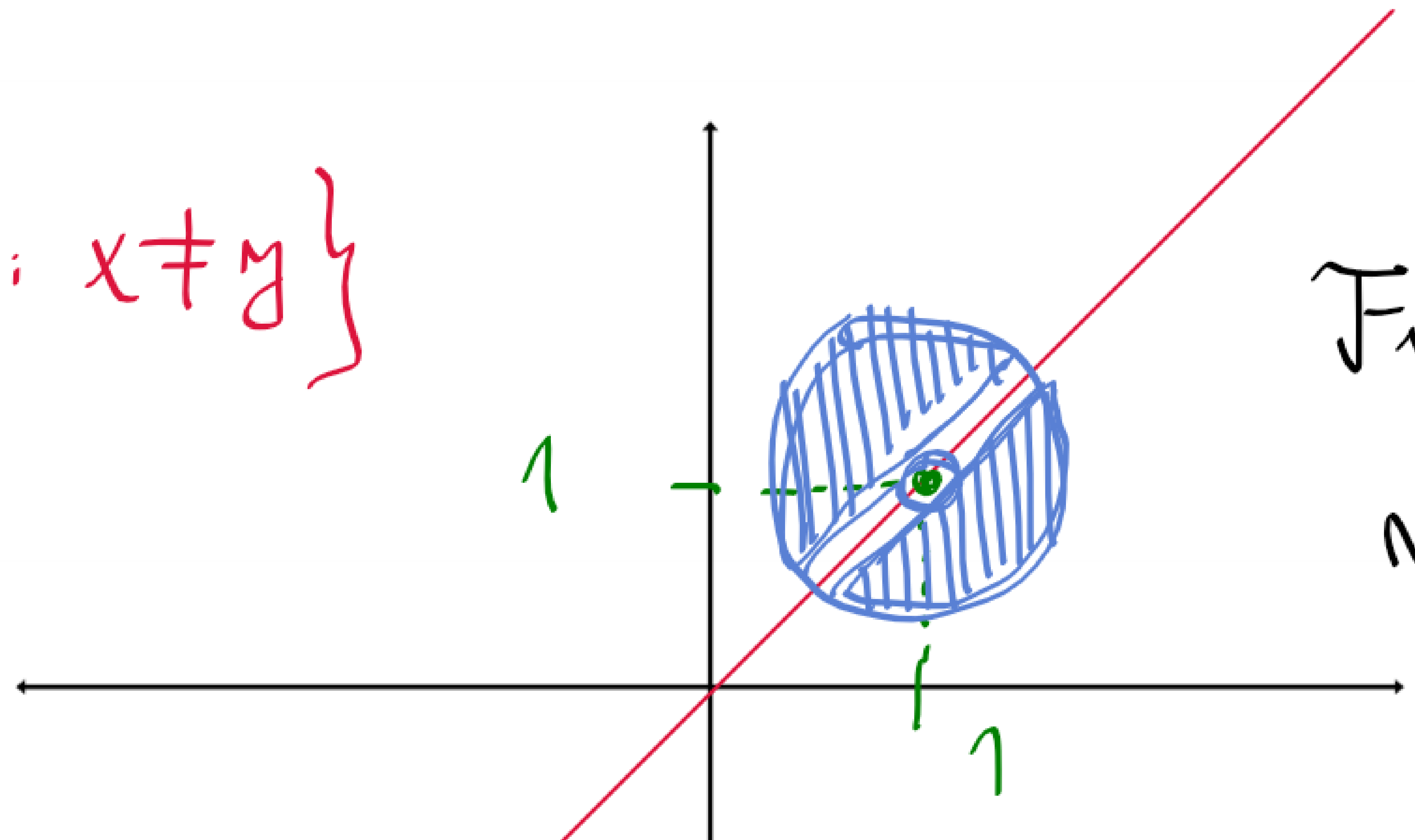
$$\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \underbrace{\frac{x^2-y^2}{x^2+3x-3y-xy}}_{f(x,y)} =$$

$$= \lim \frac{(x-y)(x+y)}{\underbrace{y(-x-3)+x(x+3)}_{x(x+3)\,-\,y(x+3)}} =$$

$$= \lim \frac{\cancel{(x-y)}(x+y)}{(x+3)\cancel{(x-y)}} = \lim \frac{\overbrace{x+y}^{\text{spoj. v } (1,1)}}{x+3} = \frac{1+1}{1+3} = \frac{1}{2}$$

$M = \{(x,y) : x \neq y\}$



Funkce f není definovaná na žádném

$P((1,1),\delta)$ ...

vždy obsahuje body mimo D_f

$$\lim_{\substack{(x,y)\to(a,b)\\ (x,y)\in M}} f(x,y) = A \quad \overset{\text{def.}}{\iff}$$

$$\forall \varepsilon>0 \ \exists \delta>0 \ \forall (x,y) \in P((a,b),\delta) \cap M$$

$$|f(x,y) - A| < \varepsilon$$

speciálně $f(x,y)$ je def.

Závěr: že bez podmínky $x \neq y$ „pod limitou" by tato lim. neexistovala.

$$\lim_{\substack{(x,y)\to(1,-3)\\ x+y+2\neq 0}} \frac{(x+y)^2-4}{x+y+2} =$$

$$= \lim \frac{(x+y-2)\cancel{(x+y+2)}}{\cancel{x+y+2}} = \underline{\underline{-4}}$$

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} \quad \dots \text{ najdeme kandidáta:}$$

$\bullet\ [y=0]:\quad \lim_{x\to 0} \frac{2x\cdot 0}{x^2+0^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$

$\bullet\ [x=0]:\quad \lim_{y\to 0} \frac{2\cdot 0\cdot y}{0^2+y^2} = \dots = 0$

$[x=y^2]:\quad \lim_{y\to 0} \frac{2y^2 y}{y^4+y^2} = \lim_{y\to 0} \frac{2y^3}{y^4+y^2} =$

$= \lim_{y\to 0} \frac{2y^{\cancel{3}}}{\cancel{y^2}(y^2+1)} = 0$

$\bullet\ [x=y]\quad \lim_{x\to 0} \frac{2x^2}{x^2+x^2} = 1 \neq 0$

$\Rightarrow$ limita $\nexists$.

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f c) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = f(x,y)$

$\bullet\ [y=0]\ \dots\ \lim_{x\to 0} \frac{x\cdot 0}{\sqrt{x^2}} = \underline{\underline{0}}.$

Řešení 1: $\bullet\ (x-y)^2 = x^2-2xy+y^2 \geq 0$

Tj. $x^2+y^2 \geq 2xy$

$\bullet\ (x+y)^2 = x^2+2xy+y^2 \geq 0$

Tj. $x^2+y^2 \geq -2xy$

Celkem: $x^2+y^2 \geq |2xy| \geq |xy|$

$$0 \le \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} \to 0$$

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} \overset{\text{spoj.}}{=} \sqrt{0^2+0^2} = 0$$

Podle LO2P $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

Řešení 2: (Heuristicky) POL. SOUŘ.:

$$x(r,\varphi) = r\cdot\cos\varphi \qquad r\in[0,\infty)$$
$$y(r,\varphi) = r\cdot\sin\varphi \qquad \varphi\in[0,2\pi)$$

$$\tilde{f}(r,\varphi) = f(x,y)$$
$$\tilde{f}(r,\varphi) = \frac{r\cos\varphi \cdot r\sin\varphi}{(r^2\cos^2\varphi + r^2\sin^2\varphi)^{1/2}} = \frac{r^2\cos\varphi\sin\varphi}{\sqrt{r^2}}$$

Tj. vlastně nás zajímá $\displaystyle \lim_{r\to0} \frac{r^2\cos\varphi\sin\varphi}{r}$

$$= \lim_{r\to0} r\cdot\cos\varphi\sin\varphi = 0$$

výsledek nezávisí na φ.

9e)
$$\lim_{(x,y)\to(0,0)} \frac{e^{(x^2+y^2)(y-2)}-1}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \overset{-2}{\overset{\uparrow}{(y-2)}}\cdot\frac{e^{(x^2+y^2)(y-2)}-1}{(x^2+y^2)(y-2)} \overset{?}{=}$$

$$-2\cdot\lim_{(x,y)\to(0,0)} \frac{e^{(x^2+y^2)(y-2)}-1}{(x^2+y^2)(y-2)} = -2\cdot 1$$

(P) $h(x,y)=0 \iff (x=y=0) \vee y=2$

$\exists\delta>0\ \forall(x,y)\in P((0,0),\delta):\ h(x,y)\neq 0$

$\delta = 2$ ✓ (melo mení)

VOLSF: $g(z) = \dfrac{e^z-1}{z}$ $\qquad \lim_{z\to0} g(z) = 1$

$h(x,y) = (x^2+y^2)(y-2)$ $\qquad \lim_{(x,y)\to(0,0)} h(x,y) = 0$

1337 a) $f(x,y) = \dfrac{\sin x + \sin y}{x+y}$

Je spojitá a def. právě na $\{(x,y) : y \neq -x\}$

Chceme spočítat $\displaystyle\lim_{\substack{(x,y)\to(a,-a)\\ y\neq -x}} f(x,y) =$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$= \lim \underbrace{2\cos\frac{x-y}{2}}\cdot\frac{\sin\frac{x+y}{2}}{\frac{x+y}{2}}\cdot\frac{1}{2} =$$

$$= 2\cos\frac{a-(-a)}{2}\,\frac{1}{2}\lim\frac{\sin\frac{x+y}{2}}{\frac{x+y}{2}} = \cos a \checkmark$$

Tedy

$$\bar f(x,y) = \begin{cases} \cos x\,, & y = -x \\ \dfrac{\sin x + \sin y}{x+y}\,, & y \neq -x \end{cases}$$

$$\lim_{(x,y)\to(a,-a)} \bar f(x,y) = \begin{cases} \displaystyle\lim_{y\neq -x} f(x,y) \\ \displaystyle\lim_{\substack{y=x\\ x\to a}} \cos x \end{cases} \cos a = \underset{\text{def.}}{\uparrow}\bar f(a,-a)$$

Tedy $\bar f$ skutečně je spojitá na $\mathbb{R}^2$.